

A RATIONAL COUNTEREXAMPLE TO THE RECIPROCAL-LOGARITHMIC ASSERTION IN ERDŐS PROBLEM 143

MATTIA APICELLA

ABSTRACT. We construct a countable set of positive rational numbers separated by at least one from all positive integer dilates of its other elements, but with divergent reciprocal-logarithmic sum. The set is built from finite primitive blocks, each equipped with a private prime. At the beginning of every round, a common integer grid makes the entire preceding configuration exclude only a small proportion of labels. Each round then adds finitely many blocks and a fixed positive amount of weight. The arithmetic input is a finite primitive-set selection lemma for periodic sets, proved here by Euler-product bounds, Dirichlet characters, and Abel averaging. All periods are fixed before the corresponding limiting argument. The construction retains every previous point and places each new point within $m^{-12}/2$ to the right of an integer m . A companion Lean 4 formalization proves the complete existence theorem, including the analytic selection lemma, all finite extensions, the recursion, and the passage to a subset of the reals with a nonsummable positive weight. Its exact statements, dependencies, and reproduction instructions are recorded.

1. THE RESULT

Write $\mathbb{N} = \{1, 2, \dots\}$ and let $\log$ denote the natural logarithm. For $x > 1$ and for a finite or countable set $B \subset (1, \infty)$, put

$$w(x) = \frac{1}{x \log x}, \quad W(B) = \sum_{x \in B} w(x) \in [0, \infty].$$

The sum is understood as the supremum of the sums over finite subsets.

Definition 1.1. A set $A \subset (1, \infty)$ is *original* if

$$|ka - b| \geq 1 \quad \text{for all } a, b \in A, \ a \neq b, \quad \text{and all } k \in \mathbb{N}. \quad (1.1)$$

The quantification is over ordered pairs, so both directions are required. A set $P \subset \mathbb{N} \setminus \{1\}$ is *primitive* if no element of P divides a distinct element of P .

In problem II on the final page of [2], Erdős asks, in particular, whether the separation condition (1.1) forces the convergence of $W(A)$. This is the reciprocal-logarithmic assertion in the problem now numbered 143 in [1]. The accompanying logarithmic-density assertion has been proved by Koukoulopoulos, Lamzouri, and Lichtman [3, Theorem 1]: every original set satisfies

$$\sum_{a \in A \cap [1, x]} \frac{1}{a} = o(\log x). \quad (1.2)$$

The theorem below disproves the convergence assertion for $W(A)$ and is consistent with (1.2).

Theorem 1.2. *There exists a countably infinite set $A \subset \mathbb{Q} \cap [2, \infty)$ with no finite accumulation point such that (1.1) holds and*

$$W(A) = \infty. \quad (1.3)$$

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Every $a \in A$ can be written as

$$a = m + \eta, \quad m \in \mathbb{N}, \quad 0 < \eta \leq \frac{1}{2}m^{-12}. \quad (1.4)$$

More precisely, the construction produces one nested sequence of finite original sets $F_1 \subset F_2 \subset \dots$ whose union is A and for which

$$W(F_R) \geq \frac{R}{576} \quad (R \in \mathbb{N}). \quad (1.5)$$

The following sections give the full argument, with its classical inputs identified explicitly. The formal theorem has no additional analytic or construction hypothesis. The companion verification is discussed in Section 8.3; independent expert review and priority are not claimed.

1.1. Outline and notation. For $n \in \mathbb{N}$, let $\Omega(n)$ be the number of prime factors of n , counted with multiplicity; in particular, $\Omega(1) = 0$. Every subset of

$$\mathcal{L}_r = \{n \in \mathbb{N} : \Omega(n) = r\}, \quad r \geq 1,$$

is primitive. Indeed, $n \mid n'$ with $n \neq n'$ implies $\Omega(n') = \Omega(n) + \Omega(n'/n) > \Omega(n)$.

A set $E \subset \mathbb{N}$ is *periodic of period L* if membership depends only on the residue modulo L , where $L \in \mathbb{N}$. Its density is

$$d(E) = \frac{\#\{b \in \mathbb{Z}/L\mathbb{Z} : b \text{ is admitted by } E\}}{L}.$$

This definition is independent of the chosen period. All density calculations below take place on a finite common period. In particular, the union bound for finitely many periodic sets is simply the union bound for finite subsets of a residue ring.

The analytic part supplies a finite primitive subset of weight greater than $1/6$ in every periodic set of density greater than $2/3$, above any fixed threshold. The geometric part uses it to construct a block of weight greater than $1/(24D)$ while spending at most $3/D$ of the density of labels on a fixed grid $D\mathbb{N}$. A round contains $\lfloor D/12 \rfloor$ blocks. Private primes make it possible to restart the next round on a new grid while preserving the entire preceding configuration.

2. PRIMITIVE WEIGHT IN A PERIODIC SET

The classical analytic inputs in this section are the absolutely convergent Euler products for $\zeta(s)$ and Dirichlet L -functions for $\Re s > 1$, character orthogonality for a fixed modulus, and continuity of $L(s, \chi)$ at $s = 1$ for nonprincipal χ ; see [4]. Non-vanishing at $s = 1$ is not needed. We use an Abel average to find arbitrarily high levels with sufficient weight, without requiring a lower bound for every sufficiently high individual level.

Theorem 2.1 (Periodic selection). *Let $E \subset \mathbb{N}$ be periodic, with $d(E) > 2/3$. For every integer $N \geq 2$, there is a finite primitive set $P \subset E \cap [N, \infty)$ such that $W(P) > 1/6$.*

For a fixed modulus $L \geq 1$ and residue $b \pmod{L}$, define

$$A_{L,b}(u) = \sum_{\substack{n \geq 2 \\ n \equiv b \pmod{L}}} u^{\Omega(n)} w(n), \quad 0 \leq u < 1. \quad (2.1)$$

We will prove convergence of this series and the sufficient estimate

$$\liminf_{u \uparrow 1} (1 - u) A_{L,b}(u) \geq \frac{1}{4L}. \quad (2.2)$$

Here and below a lower limit at least c means that, for every real $h < c$, the expression is at least h for all $u < 1$ sufficiently close to 1. The modulus and residue are fixed before the limit is taken.

The exact finite-selection statement of Theorem 2.1 is part of the complete Lean formalization by this route; the scope and reproduction of the verification are recorded in Section 8.3.

2.1. Uniform Euler-factor estimates. For a Dirichlet character χ modulo L , $0 \leq u \leq 1$, and $t > 0$, put

$$G_\chi(u, t) = \prod_p (1 - u\chi(p)p^{-1-t})^{-1} = \sum_{n \geq 1} \frac{u^{\Omega(n)} \chi(n)}{n^{1+t}}. \quad (2.3)$$

Products indexed by p are over primes. Absolute convergence follows from $|\chi(n)| \leq 1$ and $u^{\Omega(n)} \leq 1$, or directly from finite Euler products and the summability of $\sum_p p^{-1-t}$. The $n = 1$ term is 1, including when $u = 0$.

Lemma 2.2. For $0 \leq u \leq 1$ and $t > 0$,

$$|G_\chi(u, t)|^2 \leq 2|L(1 + t, \chi)|^{2u}. \quad (2.4)$$

For the principal character χ_0 modulo L , the product is real and positive and satisfies

$$G_{\chi_0}(u, t) \geq (L(1 + t, \chi_0)/2)^u. \quad (2.5)$$

Proof. For $z \in \mathbb{C}$ with $|z| \leq 1/2$, set $A = |1 - z|^2$ and $S = (1 - u) + uA$. The identity

$$|1 - uz|^2 = S - u(1 - u)|z|^2$$

and the bounds $S \geq 1/4$ and $u(1 - u) \leq 1/4$ give

$$|1 - uz|^2 \geq S(1 - |z|^2) \geq A^u(1 - |z|^2),$$

where the last step is weighted arithmetic-geometric mean. Inverting,

$$|(1 - uz)^{-1}|^2 \leq |(1 - z)^{-1}|^{2u} (1 - |z|^2)^{-1}. \quad (2.6)$$

The quadratic corrections have a uniform bound. For any finite set T of primes and $M \geq \max(T \cup \{2\})$,

$$\prod_{p \in T} (1 - p^{-2})^{-1} \leq \prod_{n=2}^M (1 - n^{-2})^{-1} = \frac{2M}{M+1} < 2. \quad (2.7)$$

Apply (2.6) with $z = \chi(p)p^{-1-t}$ and pass to the convergent Euler products to obtain (2.4).

For $0 \leq x < 1$, concavity and an elementary rearrangement give

$$(1 + x)^u \leq 1 + ux \leq (1 - ux)^{-1}.$$

For the principal character use $x = p^{-1-t}$ when $p \nmid L$ and $x = 0$ otherwise. Since $1 + x = (1 - x)^{-1}(1 - x^2)$, (2.7) implies

$$G_{\chi_0}(u, t) \geq \left(\prod_{p \nmid L} (1 + p^{-1-t}) \right)^u \geq (L(1 + t, \chi_0)/2)^u.$$

The same argument applies to finite products first; convergence then justifies the displayed passage to infinite products. $\square$

Write G_1 for the unrestricted product, corresponding to modulus one, and put

$$\kappa_L = \prod_{p|L} (1 - p^{-1}) = \frac{\varphi(L)}{L}, \quad \varphi(1) = 1.$$

Lemma 2.3. *For a fixed nonprincipal character χ ,*

$$|G_\chi(u, t)| \leq C_{L, \chi} \quad (0 \leq u \leq 1, \ 0 < t \leq 1). \quad (2.8)$$

There is $\varepsilon \in (0, 1]$, depending on L , such that

$$G_{\chi_0}(u, t) \geq \left(\frac{\kappa_L}{4t}\right)^u \quad (0 \leq u \leq 1, \ 0 < t \leq \varepsilon). \quad (2.9)$$

The unrestricted product satisfies

$$G_1(u, t) \leq 4t^{-u} \quad (0 \leq u \leq 1, \ 0 < t \leq 1). \quad (2.10)$$

Uniformly in all characters and $0 \leq u \leq 1$,

$$|G_\chi(u, t) - 1| \leq C2^{-t} \quad (t \geq 1), \quad (2.11)$$

with an absolute constant C .

Proof. For nonprincipal χ , continuity on the compact real segment $[1, 2]$ bounds $|L(s, \chi)|$. Equation (2.4) then proves (2.8), since $x^{2u} \leq \max(1, x^2)$ for $x > 0$. The principal Euler product and the elementary estimates

$$\frac{1}{t} \leq \zeta(1+t) \leq 1 + \frac{1}{t}$$

give $tL(1+t, \chi_0) \rightarrow \kappa_L$ as $t \downarrow 0$. In particular, $L(1+t, \chi_0) \geq \kappa_L/(2t)$ for small t . Equation (2.5) proves (2.9). For modulus one, (2.4) gives, when $0 < t \leq 1$,

$$G_1(u, t) \leq \sqrt{2} \zeta(1+t)^u \leq \sqrt{2} (2/t)^u \leq 4t^{-u}.$$

Finally, by the absolutely convergent series in (2.3),

$$|G_\chi(u, t) - 1| \leq \sum_{n \geq 2} n^{-1-t} \leq 2^{-(t-1)} \sum_{n \geq 2} n^{-2} \quad (t \geq 1).$$

This proves (2.11). □

2.2. Integration and the character averages.

Lemma 2.4. *For $0 \leq u < 1$, the series $\sum_{n \geq 2} u^{\Omega(n)} w(n)$ converges. For every character χ ,*

$$J_\chi(u) := \int_0^\infty (G_\chi(u, t) - 1) dt = \sum_{n \geq 2} \chi(n) u^{\Omega(n)} w(n), \quad (2.12)$$

with an absolutely convergent integral and series. Moreover,

$$C_2 := \sum_{n \geq 2} \frac{1}{n(\log n)^2} < \infty. \quad (2.13)$$

Proof. The unrestricted nonnegative integrand $G_1(u, t) - 1$ is integrable: near zero use (2.10) and $u < 1$, and near infinity use (2.11). Tonelli's theorem and the elementary identity

$$\int_0^\infty n^{-1-t} dt = \frac{1}{n \log n} \quad (n \geq 2)$$

therefore prove convergence of the unrestricted weighted series and (2.12) for the trivial character. For a general character, the sum of the integrals of the absolute values is bounded by this same convergent series. Termwise integration is consequently justified and proves (2.12).

For the last assertion, $G_1(1, t) = \zeta(1+t)$. Thus $t(\zeta(1+t) - 1)$ is bounded near zero by the elementary zeta estimates, and decays at infinity by (2.11). A further application of Tonelli, now using

$$\int_0^\infty t n^{-1-t} dt = \frac{1}{n(\log n)^2},$$

proves (2.13). $\square$

Lemma 2.5. *For each nonprincipal χ ,*

$$(1 - u)J_\chi(u) \longrightarrow 0 \quad (u \uparrow 1). \quad (2.14)$$

For the principal character,

$$\liminf_{u \uparrow 1} (1 - u)J_{\chi_0}(u) \geq \frac{\kappa_L}{4}. \quad (2.15)$$

Proof. For nonprincipal χ , equations (2.8) and (2.11) provide an integrable majorant independent of $0 \leq u < 1$ for $|G_\chi(u, t) - 1|$. Thus $J_\chi(u)$ stays bounded, which proves (2.14). For χ_0 , every term of $G_{\chi_0}(u, t) - 1$ is nonnegative. Integrating the lower bound (2.9) on $(0, \varepsilon)$ gives

$$J_{\chi_0}(u) \geq \frac{(\kappa_L/4)^u \varepsilon^{1-u}}{1 - u} - \varepsilon.$$

After multiplication by $1 - u$, the right side tends to $\kappa_L/4$. $\square$

2.3. Every residue class.

Proposition 2.6. *Estimate (2.2) holds for every fixed $L \geq 1$ and every residue $b \pmod{L}$, including noninvertible residues.*

Proof. For an invertible residue, character orthogonality and the absolute convergence in Lemma 2.4 give

$$A_{L,b}(u) = \frac{1}{\varphi(L)} \sum_{\chi \pmod{L}} \overline{\chi(b)} J_\chi(u). \quad (2.16)$$

The principal term has normalized lower limit at least $\kappa_L/(4\varphi(L)) = 1/(4L)$. The nonprincipal terms, after multiplication by $1 - u$, tend to zero by Lemma 2.5. Taking real parts in (2.16) proves the assertion. For $L = 1$, the same argument has only the principal term.

For a general residue use its representative $b \in \{0, \dots, L-1\}$ and put $g = \gcd(b, L)$, $M = L/g$, and $c = b/g$, with $\gcd(c, M) = 1$. The residue $c \pmod{M}$ is invertible, also when $M = 1$ with its unique class. Multiplication $m \mapsto gm$ sends the integers $m \geq 2$ in that class injectively into the integers $n \geq 2$ in the original class, and $\Omega(gm) = \Omega(g) + \Omega(m)$. For $m \geq 2$,

$$0 \leq \frac{1}{m \log m} - \frac{1}{m(\log m + \log g)} \leq \frac{\log g}{m(\log m)^2}. \quad (2.17)$$

Using $0 \leq u^{\Omega(m)} \leq 1$ and Lemma 2.4, we get

$$A_{L,b}(u) \geq \frac{u^{\Omega(g)}}{g} A_{M,c}(u) - \frac{\log g}{g} C_2. \quad (2.18)$$

Any possible omitted term with $m = 1$ is nonnegative, so does not affect the lower bound. Multiplication by $1 - u$ makes the error term vanish. The first coefficient tends to $1/g > 0$, and the coprime case supplies the lower bound $1/(4M) > 0$. The definition of lower limit now gives $1/(4gM) = 1/(4L)$. No coprimality between g and m is required. $\square$

2.4. Heavy levels and finite extraction.

Proof of Theorem 2.1. Fix a period L and its set S of admitted residue classes. Sum Proposition 2.6 over this finite set to obtain

$$\liminf_{u \uparrow 1} (1 - u) \sum_{\substack{n \geq 2 \\ n \in E}} u^{\Omega(n)} w(n) \geq \frac{|S|}{4L} = \frac{d(E)}{4} > \frac{1}{6}. \quad (2.19)$$

For $r \geq 0$, put

$$f_r = \sum_{\substack{n \geq 2 \\ n \in E \\ \Omega(n)=r}} w(n).$$

Each f_r is finite: the convergent weighted series at $u = 1/2$ bounds it after multiplication by 2^r . Regrouping the nonnegative summable series gives, for $0 \leq u < 1$,

$$\sum_{\substack{n \geq 2 \\ n \in E}} u^{\Omega(n)} w(n) = \sum_{r \geq 0} u^r f_r. \quad (2.20)$$

If $f_r \leq 1/6$ for all $r \geq R$, then

$$(1-u) \sum_{r \geq 0} u^r f_r \leq \frac{1}{6} + (1-u) \sum_{r < R} f_r.$$

Its upper limit would be at most $1/6$, contradicting (2.19). Thus there are arbitrarily high r with $f_r > 1/6$. Choose such an $r \geq N \geq 2$. Every integer on that level is at least $2^r \geq r \geq N$, and the level is primitive. Since its weight is a convergent positive series exceeding $1/6$, a finite truncation still exceeds $1/6$. This truncation is the required set P . $\square$

3. FINITE CONFIGURATIONS AND PRIVATE PRIMES

We index blocks by $i = 1, 2, \dots$ in their order of insertion. A block has data

$$\begin{aligned} D_i \in \mathbb{N}, \quad D_i \geq 48, \quad P_i \subset \mathbb{N} \setminus \{1\} \text{ finite, nonempty, primitive,} \\ H_i, G_i \text{ primes,} \quad p_i = H_i G_i, \quad q_i = p_i - 1, \quad c_i = \frac{p_i}{q_i}, \\ A_i = \{a_{i,n} : n \in P_i\}, \quad a_{i,n} = c_i D_i n = \frac{H_i N_{i,n}}{q_i}, \quad N_{i,n} = G_i D_i n. \end{aligned} \quad (3.1)$$

The prime H_i is called the *private prime* of the block. This terminology refers to the nondivisibility invariant below; it does not assert that H_i occurs in a reduced denominator.

Definition 3.1. A finite list of blocks is an *admissible configuration* if, writing $F = \bigcup_{i=1}^t A_i$, the following hold:

- (I1) F is original, and every point of A_i exceeds every point of $\bigcup_{j < i} A_j$ by more than 1.
- (I2) The primes H_i are distinct and greater than 2, and

$$H_j \nmid N_{i,n} \quad (1 \leq i, j \leq t, n \in P_i). \quad (3.2)$$

- (I3) Every point $a_{i,n}$ satisfies $a_{i,n}/H_i > 4$ and (1.4) with $m = D_i n$.
- (I4) The private primes satisfy the strict finite budget

$$\sum_{i=1}^t \frac{1}{H_i} < \frac{1}{100}. \quad (3.3)$$

The empty list is admissible, with $F = \emptyset$. We write $M(F) = \max F$ when $F \neq \emptyset$, and $M(\emptyset) = 0$.

All invariants concern only data that have already been selected. A new private prime is chosen large enough to preserve the remaining positive budget; no prescribed rate of prime growth is needed.

Figure 1 separates the roles of the two primes in a block. The order of choices is part of the construction: selecting the label set does not yet select the perturbation size.

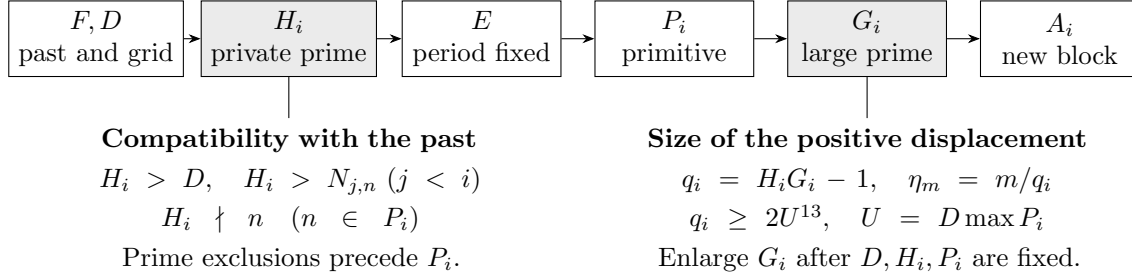


FIGURE 1. The two primes and the order of choices. Arrows indicate dependencies, not quantitative sizes. The private prime H_i enforces the nondivisibility invariant; the later choice of G_i makes the displacement small enough for all old rational sources and for the near-integer bound. No label is removed after P_i is selected.

4. A GRID COMPATIBLE WITH THE PRECEDING CONFIGURATION

Lemma 4.1. *Given an admissible configuration with point set F_0 , there exists an integer $D \geq 48$ such that*

$$\begin{aligned} N_{i,n} \mid D \quad & \text{for all preceding points,} \\ (D, H_i) = 1 \quad & \text{for all preceding private primes.} \end{aligned} \tag{4.1}$$

For this D , if $a = a_{i,n} \in F_0$, $\ell \in \mathbb{N}$, and $H_i \nmid \ell$, then

$$|ka - D\ell| > 4 \quad \text{for every } k \in \mathbb{Z}. \tag{4.2}$$

If instead $H_i \mid \ell$, then $D\ell$ is a positive integer multiple of a .

Proof. For $F_0 = \emptyset$, take $D = 48$. Otherwise let Q be the product of the distinct positive integers $N_{i,n}$ stored in F_0 . By (3.2), Q is coprime to every preceding private prime. All these primes are odd. Thus $D = 2^v Q$, with $v \geq 0$ sufficiently large, satisfies (4.1).

Fix a preceding point $a = H_i N_{i,n} / q_i$ and put $t_a = D / N_{i,n}$. This is a positive integer. Since $H_i \nmid D$ and $q_i \equiv -1 \pmod{H_i}$, we have $(t_a q_i, H_i) = 1$. The exact identity

$$|ka - D\ell| = \frac{a}{H_i} |kH_i - t_a q_i \ell| \tag{4.3}$$

holds for every $k \in \mathbb{Z}$. If $H_i \nmid \ell$, its integer factor is nonzero, so (4.2) follows from (I3). If $H_i \mid \ell$, then $D\ell/a = t_a q_i \ell / H_i$ is a positive integer. The proof uses the displayed representation of a and does not require that representation to be reduced. $\square$

For a finite rational set $F \subset (0, \infty)$ and an integer $D \geq 1$, define the set of admissible labels

$$\mathcal{R}(F, D) = \{\ell \in \mathbb{N} : ka - D\ell \notin (-1, 1] \text{ for every } a \in F \text{ and every } k \in \mathbb{Z}\}. \tag{4.4}$$

The half-open interval is chosen for the later positive perturbation.

Lemma 4.2. *The set $\mathcal{R}(F, D)$ is periodic. For $F = F_0$ and D as in Lemma 4.1,*

$$\mathcal{R}(F_0, D) = \{\ell \in \mathbb{N} : H_i \nmid \ell \text{ for every preceding } H_i\}, \tag{4.5}$$

and in particular

$$d(\mathcal{R}(F_0, D)) > \frac{99}{100}. \tag{4.6}$$

Proof. Write each $a \in F$ as A_a/B_a with positive integer numerator and denominator. Replacing ℓ by $\ell + A_a$ and k by $k + DB_a$ leaves $ka - D\ell$ unchanged. Thus the condition for this point is periodic with period A_a , and the product of these finitely many numerators is a common period. For $F = \emptyset$, every label is admitted and one may take period 1.

Equation (4.5) follows from the two alternatives in Lemma 4.1. By the Chinese remainder theorem its density is $\prod_i(1 - H_i^{-1})$, with value 1 for an empty product. The inequality $\prod_i(1 - x_i) \geq 1 - \sum_i x_i$ for $0 \leq x_i \leq 1$, together with (3.3), proves (4.6). $\square$

The reset in Lemma 4.1 changes only the grid used for future blocks. Figure 2 shows why one old source excludes just one class of labels on this grid, even though all of its integer multiples must be avoided.

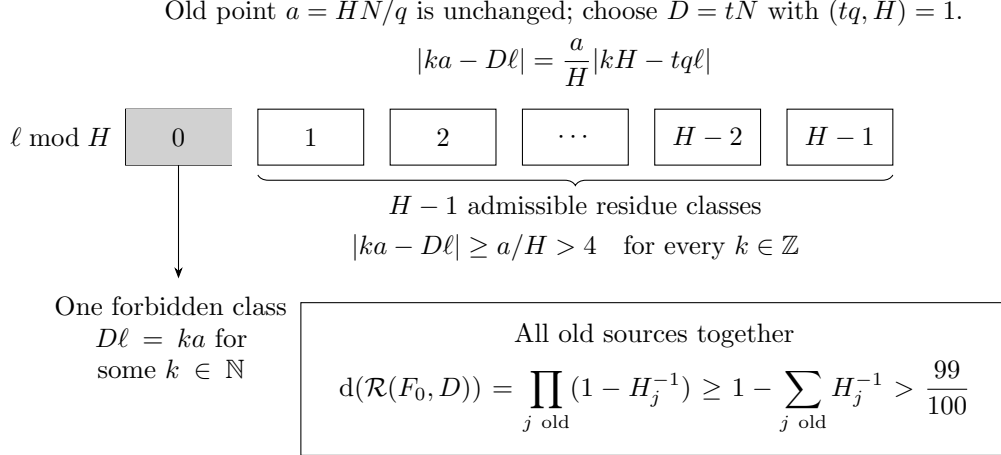


FIGURE 2. Exact residue classification after a grid reset. The strip is a symbolic view of $\mathbb{Z}/H\mathbb{Z}$; the ellipsis suppresses intermediate classes, and its drawn widths do not represent densities or distances. For a fixed old point, only $\ell \equiv 0 \pmod{H}$ causes an exact collision. The other classes have a gap greater than 4. Distinct old private primes give the product formula below the strip. All old points and their weights remain fixed.

5. THE DENSITY COST OF A BLOCK

For integers $D \geq 1$ and $p \geq 2D$, put $q = p - 1$, $c = p/q$, and $r = \lfloor q/D \rfloor$. Define a finite forbidden set of residues by

$$\mathcal{B}(p, D) = \{z \bmod p : z \in \mathbb{Z}, -r \leq z \leq r\}. \quad (5.1)$$

The interval is closed. This slightly stronger exclusion gives the half-open admissibility required in (4.4).

Lemma 5.1 (Root-residue bound). *If $\ell \bmod p \notin \mathcal{B}(p, D)$, then $|cDh - D\ell| > 1$ for every $h \in \mathbb{Z}$. Moreover,*

$$\frac{|\mathcal{B}(p, D)|}{p} \leq \frac{3}{D}. \quad (5.2)$$

Consequently the same exclusion separates $D\ell$ from every integer multiple of every point in any block $\{cDn : n \in P\}$.

Proof. If $|cDh - D\ell| \leq 1$, then $D|ph - q\ell| \leq q$. The integer $z = ph - q\ell$ therefore lies in $[-r, r]$. Since $q = p - 1$, we have $z \equiv \ell \pmod{p}$, proving the first assertion by contraposition. Reduction modulo p cannot increase cardinality, so $|\mathcal{B}(p, D)| \leq 2r + 1$. Hence

$$\frac{|\mathcal{B}(p, D)|}{p} \leq \frac{2q}{pD} + \frac{1}{p} \leq \frac{2}{D} + \frac{1}{D} = \frac{3}{D}.$$

The last estimate already holds for $p \geq D$. For a point cDn , its k -fold multiple is $cD(kn)$, so the first assertion applies with $h = kn$. No coprimality between D and p is assumed. $\square$

Remark 5.2. The number cD is a counting device only. It is not added to the original set. Its multiples contain all multiples of the entire block, which is why the bound is independent of the number of labels in P .

Fix a grid D for the configuration F_0 at the start of a round, and put

$$J = \left\lfloor \frac{D}{12} \right\rfloor. \quad (5.3)$$

After $s \leq J$ insertions, let $\mathcal{P}$ be the set of the root numerators p used in this round, so $|\mathcal{P}| \leq s$ and $p \geq 2D$ for $p \in \mathcal{P}$. Let $\mathcal{H}$ contain all current private primes, and possibly a newly chosen one, subject to $\sum_{H \in \mathcal{H}} H^{-1} < 1/100$. Set

$$E(\mathcal{H}, \mathcal{P}, D) = \left\{ \ell \in \mathbb{N} : \begin{array}{ll} H \nmid \ell & (H \in \mathcal{H}), \\ \ell \bmod p \notin \mathcal{B}(p, D) & (p \in \mathcal{P}) \end{array} \right\}. \quad (5.4)$$

This filter lies in $\mathcal{R}(F, D)$ for the entire current point set F : old-round points are protected by Lemma 4.1, and points inserted in this round by Lemma 5.1.

A common period is the product of the finitely many moduli in $\mathcal{H}$ and $\mathcal{P}$, with empty product 1. For each modulus m dividing that period, every residue modulo m has the same number of preimages. Thus lifting a forbidden set to the common period preserves its density. The finite union bound gives

$$\begin{aligned} d(E) &\geq 1 - \sum_{H \in \mathcal{H}} \frac{1}{H} - \sum_{p \in \mathcal{P}} \frac{|\mathcal{B}(p, D)|}{p} \\ &> \frac{99}{100} - \frac{3s}{D} \geq \frac{37}{50} > \frac{2}{3}. \end{aligned} \quad (5.5)$$

No independence or coprimality of these moduli is required. The single private-prime budget covers both the old-round points and all private prime exclusions; it is not counted twice.

Figure 3 plots the sufficient lower bound in (5.5). It is a bound from the proof, not a computed density for a constructed example.

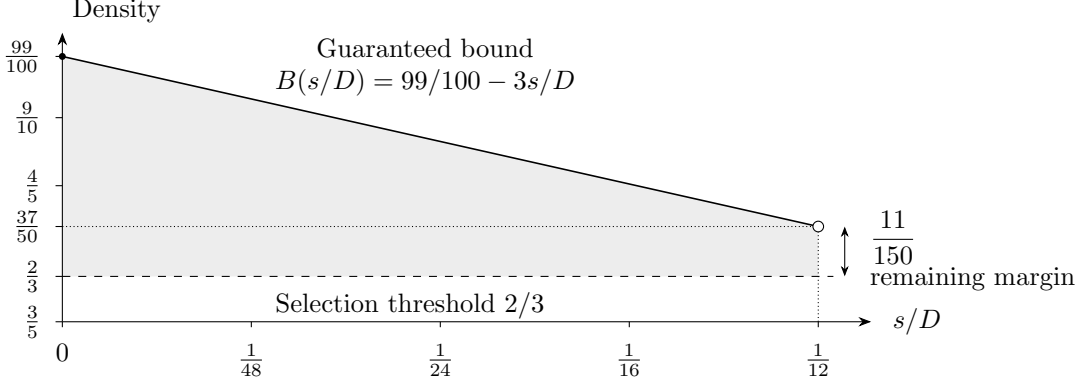


FIGURE 3. A finite-round density budget on the fixed grid $D\mathbb{N}$. The bound includes the single private-prime budget $1/100$ and at most $3s/D$ for the roots of the s blocks already inserted in this round. Actual insertions have $s < J = \lfloor D/12 \rfloor$, so $s/D < 1/12$. The open endpoint at $1/12$ is a comparison endpoint; there $B = 37/50$, still $11/150$ above the selection threshold. Shading shows this guaranteed margin, not empirical density. The vertical axis starts at $3/5$.

6. THE NEXT BLOCK

We now show that every insertion required by a round can be carried out. In addition to admissibility, assume throughout this section that the current grid D is coprime to all private primes already present. At the start of the round this follows from (4.1); new private primes in the round will be chosen greater than D .

6.1. Choosing the prime before the labels. Let i be the index of the next block and let $\delta = 1/100 - \sum_{j < i} H_j^{-1} > 0$. Choose a prime H_i exceeding

$$2, \quad D, \quad \delta^{-1}, \quad \text{all preceding } H_j, \quad \text{all preceding integers } N_{j,n}. \quad (6.1)$$

Unboundedness of the primes meets these finitely many bounds. Since $H_i^{-1} < \delta$, the enlarged private-prime set still satisfies (3.3). Let $\mathcal{H} = \{H_1, \dots, H_i\}$, let $\mathcal{P}$ contain the roots of the $s < J$ blocks already inserted in this round, and define

$$E = E(\mathcal{H}, \mathcal{P}, D). \quad (6.2)$$

The entire periodic filter, and a common period, are now fixed. By (5.5),

$$d(E) > \frac{37}{50} > \frac{2}{3}. \quad (6.3)$$

Also $E \subset \mathcal{R}(F, D)$ and every label in E avoids every current private prime, including H_i .

Apply Theorem 2.1 to this fixed E , with an integer threshold large enough to obtain a finite primitive $P_i \subset E$ with

$$\begin{aligned} W(P_i) &> \frac{1}{6}, & \min P_i &\geq 2D, \\ D \min P_i &> 4H_i, & D \min P_i &> M(F) + 1. \end{aligned} \quad (6.4)$$

Such a threshold exists because F, D, H_i are already fixed. The set P_i is nonempty, and we put $U = D \max P_i$.

6.2. Choosing the perturbation after the labels. For each $a \in F$, retain the reduced representation $a = A_a/B_a$ with positive integer denominator B_a . Choose a prime G_i , distinct from all $H_1, \dots, H_i$, sufficiently large that, with $p = H_i G_i$, $q = p - 1$, and $c = p/q$,

$$\begin{aligned} p &\geq 2D, & 1 < c < 2, & & q &\geq 2U^{13}, \\ \frac{U}{q} &< \frac{1}{B_a} & (a \in F). \end{aligned} \quad (6.5)$$

The last family is empty when $F = \emptyset$. All these conditions are finitely many lower bounds on G_i , together with finitely many prime exclusions, and so can be met simultaneously.

Set $D_i = D$ and define A_i by (3.1). For $n \in P_i$, write $m = Dn$. Then

$$a_{i,n} = cm = m + \eta_m, \quad \eta_m = \frac{m}{q} > 0, \quad (6.6)$$

and, as $m \leq U$,

$$\eta_m \leq \frac{m}{2U^{13}} \leq \frac{1}{2}m^{-12}. \quad (6.7)$$

The choice of G_i changes none of F, D, H_i, P_i .

Proposition 6.1 (Finite extension). *The enlarged configuration obtained by appending A_i is admissible. It still has $(D, H_j) = 1$ for every private prime currently present, and the new block satisfies*

$$W(A_i) > \frac{1}{24D}. \quad (6.8)$$

Proof. We first verify every type of ordered pair in the original-set condition.

An old source and a new target. Fix $a = A_a/B_a \in F$, $n \in P_i$, and $k \in \mathbb{Z}$, and set $m = Dn$. Since $n \in \mathcal{R}(F, D)$, the number $ka - m$ is outside $(-1, 1]$. It belongs to $B_a^{-1}\mathbb{Z}$, so either

$$ka - m \leq -1 \quad \text{or} \quad ka - m \geq 1 + \frac{1}{B_a}.$$

In the first case $ka - a_{i,n} = ka - m - \eta_m < -1$. In the second, (6.5) gives

$$ka - a_{i,n} \geq 1 + \frac{1}{B_a} - \eta_m > 1.$$

Thus $|ka - a_{i,n}| > 1$ for every $k \in \mathbb{Z}$, in particular for every positive k . The denominator B_a is fixed for this source, and the argument places no upper bound on k .

A new source and an old target. By (6.4) and $c > 1$, every new point $a_{i,n}$ exceeds $M(F) + 1$. For $a \in F$ and $k \in \mathbb{N}$,

$$ka_{i,n} - a \geq a_{i,n} - a > 1.$$

Two distinct points in the new block. For distinct $n, n' \in P_i$ and $k \in \mathbb{N}$, primitivity implies $kn \neq n'$. Therefore

$$|ka_{i,n} - a_{i,n'}| = cD|kn - n'| \geq cD > D > 1.$$

Old pairs already satisfy (I1). The inequalities above also give the required ordering of the new block, so (I1) holds.

For (I2), consider first a preceding integer $N_{j,n}$. The old private primes do not divide it by induction, and the new prime does not divide it because $H_i > N_{j,n}$. Next consider a new integer $N_{i,n} = G_i Dn$. For every private prime H_j with $j \leq i$:

- $H_j \nmid D$, by the round's hypothesis for $j < i$ and by $H_i > D$ for $j = i$;
- $H_j \nmid n$, by $P_i \subset E$;
- $H_j \nmid G_i$, because G_i is a prime distinct from H_j .

Since H_j is prime, it does not divide $G_i Dn$. This proves (3.2) for every pair of indices now present. The size and distinctness requirements on the private primes follow from (6.1), and the reciprocal budget was preserved when H_i was selected. No assertion about an unchosen prime is used.

For (I3), equations (6.4) and (6.7) give

$$\frac{a_{i,n}}{H_i} = \frac{cDn}{H_i} > 4, \quad 0 < a_{i,n} - Dn \leq \frac{1}{2}(Dn)^{-12}.$$

Finally, $n \geq 2D$ and $c < 2$ imply $cD \leq n$ and hence $\log(cDn) \leq 2 \log n$. Consequently,

$$w(cDn) = \frac{1}{cDn \log(cDn)} \geq \frac{w(n)}{4D}. \quad (6.9)$$

The map $n \mapsto cDn$ is injective. Summing (6.9) and using $W(P_i) > 1/6$ gives (6.8). $\square$

7. ROUNDS AND THE INFINITE UNION

Proposition 7.1 (One round). *Every admissible finite configuration can be extended, without changing any of its points or data, to an admissible finite configuration whose point set has gained weight at least $1/576$.*

Proof. Start with its point set F_0 and choose D by Lemma 4.1. Keep this D fixed, and prescribe $J = \lfloor D/12 \rfloor$ new blocks. After $s < J$ insertions, inequality (5.5) is valid. Sections 6 and 5 therefore allow the next insertion: the new configuration is admissible, its p_i satisfies $p_i \geq 2D$, and D remains coprime to every private prime. Finite induction performs all J insertions.

The blocks are pairwise disjoint by their ordering in (I1). Their total added weight is therefore, by (6.8), at least $J/(24D)$. Since $D \geq 48$, we have

$$J = \left\lfloor \frac{D}{12} \right\rfloor \geq \frac{D}{12} - 1 \geq \frac{D}{24}.$$

Thus the gain is at least $1/576$. The number of insertions is finite, even though no size bound on D is imposed. At the end of the round, (I2) allows Lemma 4.1 to choose a new grid using every integer $N_{i,n}$ now present. This new grid is used only for future blocks; no old value D_i is changed. $\square$

Proof of Theorem 1.2. Begin with the empty admissible configuration and apply Proposition 7.1 repeatedly. Let F_R be the point set at the end of round R , and set $F_0 = \emptyset$. Each round extends the same preceding configuration. Thus

$$F_0 \subset F_1 \subset F_2 \subset \cdots, \quad W(F_R) \geq \frac{R}{576} \quad (R \geq 1).$$

Set $A = \bigcup_{R \geq 1} F_R$. Every point is rational by (3.1), and every point exceeds 2, since $D_i \geq 48$ and $n \geq 2D_i$ at its insertion.

Given distinct $a, b \in A$, choose a round R with $a, b \in F_R$. The original-set property of F_R yields $|ka - b| \geq 1$ for every $k \in \mathbb{N}$. This proves (1.1) on the union, with the same pair controlling all multipliers simultaneously.

Every block is finite and nonempty, and its least element exceeds the maximum of the entire preceding prefix by more than 1. There are infinitely many blocks. Their least elements tend to infinity, so each bounded interval meets only finitely many blocks and contains only finitely many points of A . Thus A is countably infinite and has no finite accumulation point. Equation (1.4) holds for every point by (I3), and remains true because no point is ever changed.

Finally, monotonicity of nonnegative sums gives

$$W(A) \geq W(F_R) \geq \frac{R}{576} \quad \text{for every } R \geq 1.$$

Therefore $W(A) = \infty$. Equivalently, monotone convergence for this nested exhaustion gives $W(A) = \lim_{R \rightarrow \infty} W(F_R) = \infty$. $\square$

8. ORDER OF CHOICES AND FORMAL VERIFICATION

8.1. Quantifier order. The dependency order for a new block is

$$\text{preceding configuration} \longrightarrow D \longrightarrow H_i \longrightarrow E \longrightarrow P_i \longrightarrow G_i \longrightarrow A_i.$$

The grid D is chosen once per round, and is held fixed for that round. For every use of Theorem 2.1, the entire periodic set E , including all prime exclusions, is fixed before the level $\Omega(n) = r$ is chosen. Its truncation P_i is then fixed before the perturbation denominator $H_i G_i - 1$ is chosen. Thus neither a modulus uniformity assertion nor a prime exclusion imposed retroactively on P_i is part of the argument.

The private-prime induction likewise has two distinct directions. At insertion, a new H_i exceeds every old integer $N_{j,n}$; this protects all old points from the new prime. The new labels and G_i avoid all private primes already present; this protects the new points from all those primes. Repeating the same finite verification at each later stage establishes the invariant for every finite prefix.

8.2. Determinacy of the choices. The formal construction uses classical choice on the proved finite-round extension theorem, followed by recursion on the natural numbers. Only countably many successive finite choices are made. If a specified selection procedure is desired, take the least prime satisfying each finite list of bounds and the least power of two in the grid choice. For P_i , fix a period of E and an integer threshold N satisfying (6.4). Enumerate triples consisting of a level r with $2^r \geq N$, a cutoff X , and a precision for a rational lower bound on the finite sum

$$\sum_{\substack{n \leq X \\ n \in E \\ \Omega(n)=r}} \frac{1}{n \log n}.$$

Accept the first certified lower bound exceeding $1/6$. Membership in E is decidable from its finite period; for rational source points, the condition in (4.4) can also be checked by deciding whether a bounded rational interval contains an integer. Rational approximations to these finite logarithmic sums are effective. Theorem 2.1 guarantees termination of this search. No uniform bound on its runtime, the levels, the truncations, or the periods is asserted. This optional enumeration describes a mathematical selection procedure; it is not an extracted executable of the Lean construction.

8.3. Verification scope and reproducibility. The complete existence theorem is formalized in Lean 4.24.0. All names below are in the namespace `Erdos143`. The two closed rational theorems are

```
mainConclusion_proved    MainConclusion
strongConclusion_proved  StrongConclusion
```

The first gives a countable rational set in $[2, \infty)$, separation for every ordered distinct pair and every positive natural multiplier, and infinite nonnegative total weight. The second adds local finiteness and (1.4). Their definitions are in `Erdos143/Definitions.lean` and their proofs in `Erdos143/Main.lean`. Neither theorem assumes a selection principle, a finite extension, a sequence of prefixes, or any other unproved analytic or construction hypothesis.

The further theorem `realConclusion_proved`, in `Erdos143/RealConclusion.lean`, states the real-domain conclusion explicitly. It gives $A \subset \mathbb{R} \cap [2, \infty)$, countability, rationality of every

point, all the ordered multiplier inequalities, and both

$$\sum_{a \in A}^{[0, \infty]} \frac{1}{a \log a} = \infty, \quad \left(a \mapsto \frac{1}{a \log a} \right) \text{ is not summable on } A. \quad (8.1)$$

The first sum is encoded in Lean’s extended nonnegative reals `ENNReal`, with value $\top$; the second is an explicit negation of `Summable` for real-valued terms. The transfer from $\mathbb{Q}$ to $\mathbb{R}$ uses the injective rational embedding and an equivalence of the two indexing subtypes. Nonnegativity of all weights connects the two claims. Thus divergence is not represented by the conventional value of a nonsummable real `tsum`. Countable infinitude follows as well, since every finite set here has finite weight.

The analytic interface is exactly Theorem 2.1, proved by `periodicPrimitiveSelection_proved`. The principal, all-residue, and periodic Abel bounds are, respectively,

$$\begin{aligned} (2.15) \quad & \text{principal_abel_lower_bound} \\ (2.2) \quad & \text{all_residues_abel_lower_bound} \\ (2.19) \quad & \text{periodic_abel_lower_bound} \end{aligned}$$

The formal construction uses the strict finite prime budget (3.3), the closed root filter (5.1), a product reset grid, and the non-strict round gain of Proposition 7.1, as presented here. Its principal construction declarations are

Finite labels	<code>round_labels</code>
Complete insertion	<code>insert_block_exists</code>
Simultaneous invariants	<code>block_extension_valid</code>
Complete round	<code>round_extension_exists</code>
Recursive finite states	<code>stateSequence</code>
Prefix weight bound	<code>stateSequence_mass_lower</code>
Infinite total weight	<code>constructedSet_weight_diverges</code>

The formal states store finite sets of triples (H, N, q) and a finite set of private primes; the prose groups these triples into their blocks. The source notes `Abel-periodic-proof.md` and `Construction-correspondence.md` record the theorem mapping and these representational details. No lower-limit assertion for every individual Ω level is needed or claimed.

The frozen project pins `mathlib` to revision

`f897ebcf72cd16f89ab4577d0c826cd14afaafc7,`

with transitive revisions fixed in `lake-manifest.json`. On Linux x86_64, reproduction from the supplied source archive is:

```
bash scripts/bootstrap.sh
bash scripts/verify.sh
```

The bootstrap retrieves the pinned official Lean release and `mathlib` dependencies; it requires network access and the tools listed in the project’s `README.md`. The verification builds with warnings as errors, checks the sources for proof placeholders, audits all proved declarations, and prints the expanded existential statements. Two local verification runs succeeded, including a clean rebuild of the project from the frozen sources. The source hashes agreed before and after the second run. These were local computational checks, not external expert reviews.

All 245 proved declarations passed the axiom audit. In particular, the three closed conclusion theorems depend only on `propext`, `Classical.choice`, and `Quot.sound`: propositional extensionality, classical choice, and quotient soundness. There are no additional analytic or arithmetic axioms and no proof placeholders.

The verification record and both sets of local logs are included with the frozen source archive

`erdos143-lean-complete-verified-20261002.zip`.

Its SHA-256 digest, split across two lines only for typesetting, is

d4a8477e5122ad7f66584483b84a93b1
4af38b3a412c8b9378617b2c2137fec6

The article’s LaTeX archive is a separate typesetting artifact; compiling it does not run the formalization. Kernel checking establishes the formal statements from the recorded foundations. Independent expert review of the mathematical formulation, the correspondence to Erdős’s question, and the exposition has not yet been completed. No claim of priority, external acceptance, or publication is made.

REFERENCES

- [1] T. F. Bloom, *Erdős problem #143*, Erdős Problems, <https://www.erdosproblems.com/143>, consulted October 2, 2026.
- [2] P. Erdős, *On the density of some sequences of integers*, Bull. Amer. Math. Soc. **54** (1948), 685–692, especially problem II, p. 692. doi:10.1090/S0002-9904-1948-09057-7. Author archive copy.
- [3] D. Koukoulopoulos, Y. Lamzouri, and J. D. Lichtman, *Erdős’s integer dilation approximation problem and GCD graphs*, preprint, February 13, 2025, Theorem 1. Author-hosted manuscript.
- [4] T. Tao, *254A, Notes 1: Elementary multiplicative number theory*, What’s New, November 23, 2014. Online lecture notes.